Financial Econometrics A | Final Exam | January 2nd, 2018

Solution Key

Question A:

Consider the model for $x_t \in \mathbb{R}$ given by

$$x_t = \sqrt{1 + \beta x_{t-1}^2} z_t \tag{A.1}$$

where the innovation z_t satisfies

$$z_t \sim i.i.d.N(0, \omega). \tag{A.2}$$

The model parameters $\theta = (\beta, \omega)$ satisfy $\beta \geq 0$ and $\omega > 0$.

Question A.1: Provide conditions on $\theta = (\beta, \omega)$ such that x_t satisfies the drift criterion with drift function $\delta(x) = 1 + x^2$.

Solution: It can be noted that $x_t = \sqrt{\omega + \omega\beta x_{t-1}^2}\tilde{z}_t$ with $\tilde{z}_t \sim i.i.d.N(0, 1)$. Hence, standard derivations from the lecture note yield that the drift criterion is satisfied if $\omega\beta < 1$. Detailed arguments should be provided, including that x_t is a Markov chain with a positive and continuous transition density.

Question A.2: The log-likelihood contribution for the model is

$$l_t(\theta) = -\frac{1}{2}\left[\log(\omega + \omega\beta x_{t-1}^2) + \frac{x_t^2}{\omega + \omega\beta x_{t-1}^2}\right]. \tag{A.3}$$

With $\theta_0 = (\beta_0, \omega_0)$ the true value of θ, suppose that x_t is weakly mixing such that $E[x_t^2] < \infty$. Argue that for some constant $c > 0$,

$$E\left[\frac{x_t^2}{\omega_0 + \omega_0\beta_0 x_{t-1}^2}\right] \leq c.$$

Solution: We note that

$$E\left[\frac{x_t^2}{\omega_0 + \omega_0\beta_0 x_{t-1}^2}\right] = E\left[\frac{(1 + \beta_0 x_{t-1}^2)z_t^2}{\omega_0 + \omega_0\beta_0 x_{t-1}^2}\right] = E\left[\frac{z_t^2}{\omega_0}\right] = 1 =: c$$

Alternatively, one may note that $1/(\omega_0 + \omega_0\beta_0 x_{t-1}^2) \leq \omega_0^{-1}$, such that

$$E\left[\frac{x_t^2}{\omega_0 + \omega_0\beta_0 x_{t-1}^2}\right] \leq E\left[\frac{x_t^2}{\omega_0}\right] = \omega_0^{-1}E\left[x_t^2\right] =: c,$$

using that $E[x_t^2] < \infty$.

Question A.3: Let $s_t(\theta)$ denote the first derivative in the direction ω of the log-likelihood contribution in (A.3), i.e.

$$s_t(\theta) = \frac{\partial l_t(\theta)}{\partial \omega}.$$

Let $\theta_0 = (\beta_0, \omega_0)$ be the vector of true parameter values. Note that $E[z_t^4] = 3\omega_0^2$.
With T the sample size, provide conditions such that as $T \to \infty$,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} s_t(\theta_0) \xrightarrow{D} N(0, \Sigma), \quad \Sigma = \frac{1}{2\omega_0^2} > 0. \tag{A.4}$$

Explain what (A.4) can be used for.

Solution: The result in (A.4) can established by verifying the conditions of the CLT for weakly mixing processes (Theorem II.1). Straightforward derivations yield that

$$s_t(\theta_0) = \frac{1}{2\omega_0} \left(\frac{z_t^2}{\omega_0} - 1 \right) =: f(x_t, x_{t-1}).$$

We note that $E[f(x_t, x_{t-1})|x_{t-1}] = 0$ and $E[f^2(x_t, x_{t-1})] < \infty$. *Assuming that x_t is weakly mixing* (cf. Question A.1), the CLT implies that

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{T} s_t(\theta_0) \xrightarrow{D} N(0, \Sigma),$$

with $\Sigma = E[f^2(x_t, x_{t-1})]$. Note that

$$E[f^2(x_t, x_{-1})] = E\left[\frac{1}{4\omega_0^2} \left(\frac{z_t^2}{\omega_0} - 1 \right)^2 \right] = \frac{1}{4\omega_0^2} E\left[\frac{z_t^4}{\omega_0^2} + 1 - 2\frac{z_t^2}{\omega_0} \right] = \frac{1}{2\omega_0^2}.$$

Derivations should be provided.
One may note that $s_t(\theta_0)$ is an i.i.d. process, so that (A.4) holds by a CLT for i.i.d. processes. Hence the assumption that x_t is weakly mixing is redundant. The latter remark is for sure not required.
The property (A.4) is used for deriving the limiting distribution of the MLE. Ideally, a few comments about this is included. The distribution of the MLE is used when testing a hypothesis about the model parameters.

Question A.4: For the model (A.1)-(A.2), the one-period Value-at-Risk (VaR) at risk level κ, $\text{VaR}^{\kappa}_{T,1}$, is

$$\text{VaR}^{\kappa}_{T,1} = -\omega^{1/2}\sigma_{T+1}\Phi^{-1}(\kappa), \quad \kappa \in (0,1),$$

where $\sigma^2_{T+1} = 1 + \beta x^2_T$ and where $\Phi^{-1}(\cdot)$ denotes the inverse CDF of the standard normal distribution.
Explain briefly how you would compute an estimate of $\text{VaR}^{\kappa}_{T,1}$.
Explain briefly how you would compute an estimate of the two-period VaR at risk level κ.

Solution: Given some estimate of (ω, β), denoted $(\hat{\omega}, \hat{\beta})$, we obtain an estimate of $\text{VaR}^{\kappa}_{T,1}$ as

$$-\hat{\omega}^{1/2}(1 + \hat{\beta}x^2_T)^{1/2}\Phi^{-1}(\kappa),$$

where we note that $\Phi^{-1}(\kappa)$ is known.
We do not have a closed-form expression for the two-period VaR in term of x_T and (ω, β). Instead, this quantity is typically estimated by filtered historical simulation (FHS). Ideally, a brief outline of this should be provided.

Question B:

Consider the model for $x_t \in \mathbb{R}$ given by

$$x_t = a_{s_t} x_{t-1} + \varepsilon_t, \tag{B.1}$$

where the error term ε_t satisfies

$$\varepsilon_t \sim i.i.d.N(0,1). \tag{B.2}$$

Moreover,

$$a_{s_t} = 1(s_t = 1)a_1 + 1(s_t = 2)a_2, \tag{B.3}$$

where s_t is a state variable that takes values in $\{1,2\}$ according to the transition probabilities

$$P(s_t = j | s_{t-1} = i) = p_{ij}, \tag{B.4}$$

and $1(s_t = i) = 1$ if $s_t = i$ and $1(s_t = i) = 0$ if $s_t \neq i$ for $i = 1, 2$. We assume throughout that the processes (ε_t) and (s_t) are independent. The model parameters $\theta = (a_1, a_2)$ satisfy $a_1, a_2 \in \mathbb{R}$.

Question B.1: When is the process (s_t) weakly mixing?

Solution: It is well-known that s_t is weakly mixing if $p_{11}, p_{22} < 1$ (irreducibility) and $p_{11} + p_{22} > 0$ (aperiodicity).

Question B.2: Let $f(x_t|x_{t-1}, s_t)$ denote the conditional density of x_t given (x_{t-1}, s_t). Provide an expression for $f(x_t|x_{t-1}, s_t)$.

Solution: For $i = 1, 2$, we note that $(x_t|x_{t-1}, s_t = i) \sim N(a_i x_{t-1}, 1)$. Hence, for $i = 1, 2$,

$$f(x_t|x_{t-1}, s_t = i) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x_t - a_i x_{t-1})^2}{2} \right).$$

We conclude that

$$f(x_t|x_{t-1}, s_t) = \prod_{i=1}^{2} f(x_t|x_{t-1}, s_t = i)^{1(s_t = i)}.$$

Question B.3: In the following we assume that $p_{11} = 1 - p_{22} =: p \in (0,1)$ such that (s_t) is an i.i.d. process with $P(s_t = 1) = p$.
Show that

$$f(x_t|x_{t-1}) = f(x_t|x_{t-1}, s_t = 1)P(s_t = 1) + f(x_t|x_{t-1}, s_t = 2)P(s_t = 2) > 0.$$

Show that x_t satisfies the drift criterion with drift function $\delta(x) = 1 + x^2$ if

$$a_1^2 p + a_2^2(1-p) < 1.$$

Solution: Using that the events $\{s_t = 1\}$ and $\{s_t = 2\}$ are disjoint, and that s_t and x_{t-1} are independent,

$$\begin{aligned} f(x_t|x_{t-1}) &= f(x_t, s_t = 1|x_{t-1}) + f(x_t, s_t = 2|x_{t-1}) \\ &= f(x_t|s_t = 1, x_{t-1})P(s_t = 1|x_{t-1}) + f(x_t|s_t = 2, x_{t-1})P(s_t = 2|x_{t-1}) \\ &= f\left(x_t|x_{t-1}, s_t = 1\right) P\left(s_t = 1\right) + f\left(x_t|x_{t-1}, s_t = 2\right) P(s_t = 2). \end{aligned}$$

We note that x_t is a Markov chain with positive and continuous transition density, $f(x_t|x_{t-1})$. Moreover,

$$\begin{aligned} &E[1 + x_t^2|x_{t-1}] \\ &= 1 + E[(1(s_t = 1)a_1 + 1(s_t = 2)a_2)^2 x_{t-1}^2 + \varepsilon_t^2 + 2\left(1(s_t = 1)a_1 + 1(s_t = 2)a_2\right) x_{t-1}\varepsilon_t|x_{t-1}] \\ &= 2 + E[(1(s_t = 1)a_1 + 1(s_t = 2)a_2)^2 x_{t-1}^2|x_{t-1}] \\ &= 2 + a_1^2 E[1(s_t = 1)|x_{t-1}]x_{t-1}^2 + a_2^2 E[1(s_t = 2)|x_{t-1}]x_{t-1}^2 \\ &= 2 + \left[a_1^2 P(s_t = 1) + a_2^2 P(s_t = 2)\right] x_{t-1}^2 \\ &= 2 + \left[a_1^2 p + a_2^2(1-p)\right] x_{t-1}^2. \end{aligned}$$

Usual derivations yield that the drift criterion is satisfied, if $[a_1^2 p + a_2^2(1-p)] < 1$. Details should be provided.

Question B.4: Maintaining the assumptions from Question B.3, we consider the log-likelihood function (up to a constant)

$$\begin{aligned} L_T(\theta) = \sum_{t=1}^{T} \Bigg[& 1\left(s_t = 1\right) \left\{ -\frac{(x_t - a_1 x_{t-1})^2}{2} + \log(p) \right\} \\ & + 1(s_t = 2) \left\{ -\frac{(x_t - a_2 x_{t-1})^2}{2} + \log(1-p) \right\} \Bigg]. \end{aligned}$$

Show that the Maximum Likelihood Estimator for a_1 is

$$\hat{a}_1 = \frac{\sum_{t=1}^{T} 1(s_t = 1) x_t x_{t-1}}{\sum_{t=1}^{T} 1(s_t = 1) x_{t-1}^2}.$$

Assume that the joint process (s_t, x_{t-1}) is weakly mixing such that $E[x_{t-1}^2] < \infty$. Argue that $\hat{a}_1 \xrightarrow{P} a_1$ as $T \to \infty$.

Solution: Solving $\partial L_T(\theta)/\partial a_1 = 0$ yields $\hat{a}_1$. Moreover,

$$\begin{aligned}\hat{a}_1 &= \frac{\sum_{t=1}^T 1(s_t = 1)[(1(s_t = 1)a_1 + 1(s_t = 2)a_2)x_{t-1} + \varepsilon_t]x_{t-1}}{\sum_{t=1}^T 1(s_t = 1)x_{t-1}^2} \\ &= a_1 + \frac{T^{-1}\sum_{t=1}^T 1(s_t = 1)\varepsilon_t x_{t-1}}{T^{-1}\sum_{t=1}^T 1(s_t = 1)x_{t-1}^2}.\end{aligned}$$

By the LLN for weakly mixing process (using that $E[x_t^2] < \infty$),

$$T^{-1}\sum_{t=1}^T 1(s_t = 1)\varepsilon_t x_{t-1} \xrightarrow{P} E[1(s_t = 1)\varepsilon_t x_{t-1}] = 0$$

and

$$T^{-1}\sum_{t=1}^T 1(s_t = 1)x_{t-1}^2 \xrightarrow{P} E[1(s_t = 1)x_{t-1}^2] = pE[x_{t-1}^2] < \infty.$$

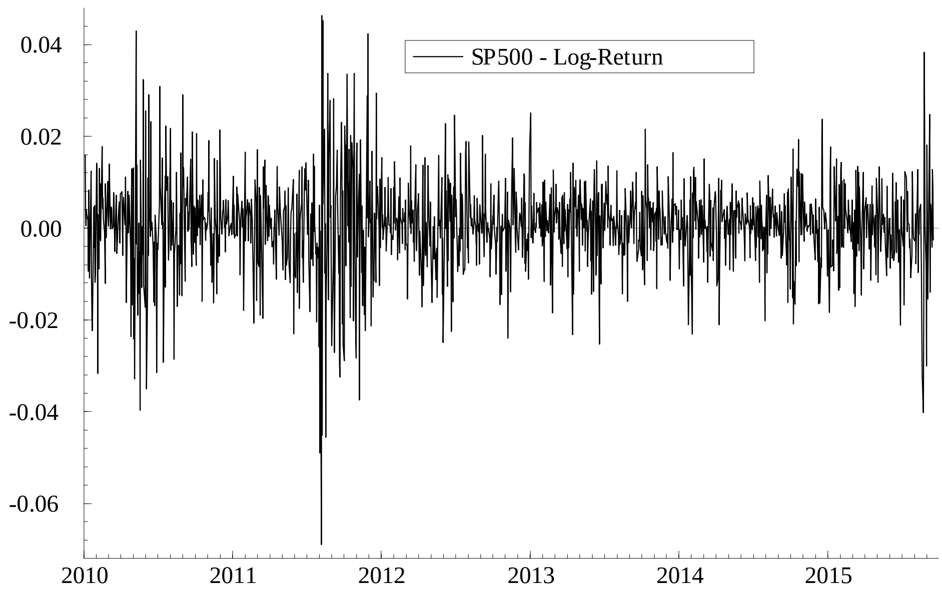
We conclude that $\hat{a}_1 \xrightarrow{P} a_1$.

Question B.5: The following figure shows the daily log-returns of the S&P 500 index for the period January 4, 2010 to September 17, 2015.
Discuss briefly whether the model in (B.1)-(B.4) is a reasonable model for the daily log returns of the S&P 500 index.

Solution: Clearly, and as discussed during lectures, the log-returns appear to exhibit volatility clustering. The model has time-varying conditional mean, whereas $V(x_t|x_{t-1}, s_t = i) = 1$ for $i = 1, 2$, and hence the conditional variance of x_t (given the state) is constant. Hence the model does not appear suitable for describing the main feature of the data series.
A more rigorous (although not required) answer may note that $a_{s_t} = (a_1 - a_2)1(s_t = 1) + a_2$ such that

$$V(x_t|x_{t-1}) = 1 + (a_1 - a_2)^2 x_{t-1}^2 V[1(s_t = 1)|x_{t-1}],$$

where $V[1(s_t = 1)|x_{t-1}] = P(s_t = 1|x_{t-1})[1 - P(s_t = 1|x_{t-1})]$, suggesting that $V(x_t|x_{t-1})$ is timevarying for $a_1 \neq a_2$.


SP500 - Log-Return
0.04
0.02
0.00
-0.02
-0.04
-0.06
2010
2011
2012
2013
2014
2015